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ABSTRACT

This thesis analyses the effect spin-orbit coupling has on the dispersion of Langmuir waves in magnetized plasmas, using recently developed kinetic theories of plasmas including quantum mechanical and relativistic effects. Two new wave modes appear close to the resonance $\Delta\omega_c = (g/2-1)\omega_c$, where ω_c is the cyclotron frequency and g is the electron gyromagnetic ratio. For considered long wave lengths the deviation from this resonance is very small. The wave modes are also very weakly damped.

CONTENTS

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OVERVIEW

Considering recent developments of kinetic models for plasmas along with the knowledge of how to generalize these models to include spin-orbit coupling, interesting new problems to study arise. In this thesis we started from a semi-classical and semi-relativistic kinetic theory of plasmas and examined how the dispersion of Langmuir waves propagating in a magnetized plasma is affected by the inclusion of spin-orbit coupling. The method we used was to simplify a Lorentz invariant evolution equation by linearizing it and observe how the motion of the particles follow the oscillations of Langmuir waves. Maxwell's equations then allowed us to solve the problem and find the dispersion relation. From this dispersion relation new wave modes were expected to appear.

We will start by briefly introducing the concepts and theories used.

1.1 INTRODUCTION TO PLASMAS

Plasmas are popularly termed as the fourth state of matter along-side gases, fluids and solids. As defined by D. R. Nicholson [6]: *"A* plasma *is a gas of charged particles, in which the potential energy of a typical particle due to its nearest neighbour is much smaller than its kinetic energy."*

Having a high kinetic energy often (but not necessarily!) coincides with having a high temperature as well as electrons having a high enough energy to break free from its host atom, plasmas hence often consist of mostly ionized atoms. A comparably low potential energy due to neighbouring particles means that collective effects of all particles dominate over local effects in small concentrations of particles. For instance, plasmas exhibit a phenomenon known as *Debye shielding*, that small concentrations of charges or potentials are shielded from electric fields of collective particles as nearby particles rush in to negate the effect. Thanks to this shielding, collisions in plasmas generally have a weak effect which is what we will consider for our theories in this paper.

Considered to make up more than 99 % of all matter in the known universe (excluding the recently discovered dark matter), examples of plasmas are mainly found in astrophysics, such as stars, solar winds and nebulas. On the earth the element can be seen in for example lightning strikes, aurora borealis and plasma displays.

1.2 LANGMUIR WAVES

Langmuir waves, which are studied in this thesis, are by definition *plasma oscillations*. That is, if a charge seperation occurs in a plasma the particles will react to the electric field and begin to oscillate around a point of equilibrium [3]. The waves are electrostatic by nature and propagate along an external magnetic field. Using Maxwell's laws it is easy to see that they hence do not modify or create any magnetic field.

1.3 SPIN-ORBIT COUPLING

Spin-orbit coupling is a relativistic effect arising as the particles move in an electric field: As they move in the field they feel a magnetic field in their rest frame as per the theory of special relativity and this magnetic field interracts with the spin [5]. In, for instance, atomic and molecular theories this coupling modifies the energies of particles considerably, giving rise to a fine structure of energy levels [4].

1.4 SEMI-CLASSICAL SPIN

A semi-classical theory of spin will be considered in this work, wherein spin is described as a classic vector on the unit sphere. This picture is intrinsically wrong since a spin can not ever point in any one direction, we can however use a distribution function of the probability density to measure a spin in a certain direction. For a given classic spin vector the distribution function then yields the probability to measure the spin in that direction.

2 KINETIC THEORY

The basis of this thesis is the kinetic theory of plasmas: A description that is intuitive, contains all the classic effects of plasmas (for instance, Landau damping) and can be modified to include quantum mechanical and relativistic effects. This chapter gives a brief introduction to the theory and how it has been modified to include the effects we are studying.

2.1 CLASSICAL THEORY

The classic distribution function $f(\vec{x}, \vec{v}, t)$ gives the probability density to find a particle at position $\vec{x}$ with velocity $\vec{v}$ at time t. An equation for how this distribution changes over time can be derived exactly from the Klimontovich or Liouville equation [6], but for our purposes a simple motivation will be sufficient. If considering a single particle in phase-space, at the location of the particle the probability to find it is always one. Hence along the particle path the value of the distribution function does not change:

$$0 = \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{d\vec{v}}{dt} \cdot \frac{\partial f}{\partial \vec{v}} \tag{2.1}$$

As the acceleration is due to the Lorentz force for a particle in an electric field $\vec{E}$ and magnetic field $\vec{B}$, we have

$$\frac{d\vec{x}}{dt} = \vec{v} \tag{2.2}$$

$$\frac{d\vec{v}}{dt} = \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}), \tag{2.3}$$

where q denotes the charge of the particle and m the mass. Thus we get

$$0 = \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{v}} \tag{2.4}$$

which is the Vlasov equation, or collision-free Boltzmann equation. As implied by the motivation we used to derive it, it describes how an ensemble of particles in a plasma (where individual effects as collisions are less important than the collective effects) evolves with time under applied electromagnetic fields. The equation is consistent with quantum mechanics considering sufficiently long scale lengths.

For this system, our electromagnetic sources are the charge distribution ρ and the free current density $\vec{j}_F$, both of which can be described using the distribution function $f(\vec{x}, \vec{v}, t)$:

$$\rho = q \int d^3v\, f(\vec{x}, \vec{v}, t) \tag{2.5}$$

$$\vec{j}_F = q \int d^3v\, \vec{v}\, f(\vec{x}, \vec{v}, t) \tag{2.6}$$

These expressions can be grasped intuitively by considering that the integral $\int d^3v\, f(\vec{x}, \vec{v}, t)$ gives the particle density n_0 at position $\vec{x}$ and time t, which means that qn_0 is the charge density at those points. The integral $\int d^3v\, \vec{v}\, f(\vec{x}, \vec{v}, t)$ on the other hand gives an overall velocity $\vec{u}$ of the particle density n_0 and $qn_0\vec{u}$ is hence the current density.

2.2 QUANTUM MECHANICAL EFFECTS

Adding quantum effects to this equation will for our purposes be done using a semi-classical spin vector $\vec{s}$ and adding this as another independent variable in the distribution function: $f(\vec{x}, \vec{v}, \vec{s}, t)$ (it can however also be derived fully quantum mechanically using the density matrix and phase-space distribution functions [9]). Using the same motivation as before, a spin dependent term is added to eq. (2.1)

$$0 = \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{d\vec{v}}{dt} \cdot \frac{\partial f}{\partial \vec{v}} + \frac{d\vec{s}}{dt} \cdot \frac{\partial f}{\partial \vec{s}}. \tag{2.7}$$

In the quantum mechanical case, the time evolution of the velocity and spin are calculated from the Pauli Hamiltonian as [2],

$$\frac{d\vec{v}}{dt} = \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) + \frac{\mu}{m}\frac{\partial}{\partial \vec{x}}(\vec{s} \cdot \vec{B}) \tag{2.8}$$

$$\frac{d\vec{s}}{dt} = \frac{2\mu}{\hbar}(\vec{s} \times \vec{B}), \tag{2.9}$$

where μ is the magnetic moment of the particle and $\hbar$ is Planck's reduced constant. The second term in eq. (2.8) is coupled to the spin dipole moment working to minimize its potential in a changing magnetic field, while the term in eq. (2.9) describes a spin precessing around the magnetic field. Lastly, a quantum mechanical effect not explained by the semi-classical theory is added to the equation,

$$\frac{\mu}{m}\frac{\partial}{\partial \vec{x}} \cdot \left[\left(\vec{B} \cdot \frac{\partial}{\partial \vec{s}} \right) \frac{\partial f}{\partial \vec{v}} \right]. \tag{2.10}$$

This term compensates for the fact that all components of the spin can not be known simultaneously [9].

Everything included, we get a quantum mechanical, non-relativistic evolution equation as

$$\begin{aligned} 0 = \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \left[\frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) + \frac{\mu}{m} \frac{\partial}{\partial \vec{x}}(\vec{s} \cdot \vec{B}) \right] \cdot \frac{\partial f}{\partial \vec{v}} \\ + \frac{2\mu}{\hbar}(\vec{s} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{s}} + \frac{\mu}{m} \frac{\partial}{\partial \vec{x}} \cdot \left[\left(\vec{B} \cdot \frac{\partial}{\partial \vec{s}} \right) \frac{\partial f}{\partial \vec{v}} \right] \end{aligned} \quad (2.11)$$

which is valid in the long scale length limit, where the potential changes are small over the distance of the thermal de Broglie wave length $\hbar/mv_t$ and v_t is the thermal velocity of the particle.

The magnetization from the particle spin also gives an extra contribution to the current density in addition to the free current, as [2]

$$\vec{j}_{\mathrm{M}} = \nabla \times 3\mu \int d\Omega\, \vec{s} f(\vec{x}, \vec{v}, \vec{s}, t) \quad (2.12)$$

where the factor 3 is to account for the spin never pointing in any one direction but being smeared out over the surface of a sphere and $d\Omega = d^3v\, d^2s$ is the complete integration element in our phase-space.

2.3 RELATIVISTIC EFFECTS

Now modifying the quantum mechanical evolution equation to create a Lorentz invariant theory gives us [1]

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} \\ &+ \left\{ \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) + \frac{\mu}{m} \frac{\partial}{\partial \vec{x}} \left[\vec{s} \cdot \left(\vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right) \right] \right\} \cdot \frac{\partial f}{\partial \vec{v}} \\ &+ \frac{2\mu}{\hbar} \left[\vec{s} \times \left(\vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right) \right] \cdot \frac{\partial f}{\partial \vec{s}} \\ &+ \frac{\mu}{m} \frac{\partial}{\partial \vec{x}} \cdot \left\{ \left[\left(\vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right) \cdot \frac{\partial}{\partial \vec{s}} \right] \frac{\partial f}{\partial \vec{v}} \right\}, \end{aligned} \quad (2.13)$$

where c is the speed of light in vacuum and the spin-orbit coupling terms are present. These terms appear as the additional magnetic fields $\vec{B} = -(\vec{v} \times \vec{E})/c^2$ that the particles perceive in their rest frame as they move in another frame of reference [5]. Containing all the effects we wish to study in this project, this thus is the evolution equation that will serve as a basis for our calculations. Note that while this argument gives the right result, it is not as rigorous as it seems – a more careful derivation of the transforming properties of all variables must be made.

As with the evolution equation, the current density will also be affected when considering effects of special relativity as [1]

$$\vec{j}_{\mathrm{P}} = -\frac{3\mu}{c^2} \frac{\partial}{\partial t} \int d\Omega\, \vec{v} \times \vec{s}\, f(\vec{x}, \vec{v}, \vec{s}, t)\,. \quad (2.14)$$

This added polarization current can be seen as a relativistic correction of the electric field for the moving magnetization from eq. (2.12). We will give a brief motivation of this result, which can be derived more explicitly using covariant tensor notation: Ampere's law is given as

$$\nabla \times \vec{B} = \mu_0 \vec{j}_{\text{tot}} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}, \tag{2.15}$$

where the total current density is given by $\vec{j}_{\text{tot}} = \vec{j}_{\text{F}} + \vec{j}_{\text{M}} + \vec{j}_{\text{P}}$. The non-free current densities can be written as

$$\vec{j}_{\text{M}} = \nabla \times \vec{M} \tag{2.16}$$

$$\vec{j}_{\text{P}} = \frac{\partial \vec{P}}{\partial t} \tag{2.17}$$

for a *magnetization* $\vec{M}$ and *polarization* $\vec{P}$ [5]. Ampere's law can then be rewritten as

$$\nabla \times \left(\vec{B} - \mu_0 \vec{M}\right) = \mu_0 \vec{j}_{\text{F}} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{E} + \frac{\vec{P}}{\epsilon_0}\right). \tag{2.18}$$

Noting the similarities between the magnetic and electric fields and the magnetization and polarization, we can see that they should transform identically into other frames of references (this can be shown explicitly using the electromagnetic field tensor and four-current on the same form). There is no electric dipole moment in the rest frame of the particles and hence no contributions to the polarization. Thus the magnetization and polarization transform to first order as [5]

$$\vec{M} \to \vec{M} \tag{2.19}$$

$$\vec{P} \to \frac{\vec{v} \times \vec{M}}{c^2}. \tag{2.20}$$

Inserting this polarization into eq. (2.17) we thus see that a moving magnetization creates an additional current density as in eq. (2.14), which is what we wanted to show.

Finally collecting all terms gives us the total current density in our theory as

$$\begin{aligned}\vec{j}_{\text{tot}} = q \int d^3v\, \vec{v}\, f(\vec{x}, \vec{v}, \vec{s}, t) + \nabla \times 3\mu \int d\Omega\, \vec{s}\, f(\vec{x}, \vec{v}, \vec{s}, t) \\ - \frac{3\mu}{c^2}\frac{\partial}{\partial t}\int d\Omega\, \vec{v} \times \vec{s}\, f(\vec{x}, \vec{v}, \vec{s}, t)\end{aligned} \tag{2.21}$$

which is the equation we will use.

3 THE DISPERSION RELATION

To find the dispersion relation for our electrostatic waves our employed method is to linearize and Fourier analyse the kinetic theory with relativistic corrections, calculate the distribution function and total current density and finally use Ampere's law (2.15)

$$\nabla \times \vec{B} = \mu_0 \vec{j}_{\text{tot}} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

to close the system.

3.1 LINEARIZING THE EVOLUTION EQUATION

When studying the linear modes of the relativistic evolution equation (2.13) we are assuming that all oscillations of particle densities and fields are small in order to simplify it. We are considering a distribution of particles f such that one term contains a static background and one term the oscillations. We are only considering electron oscillations by picking a suitable time-scale such that the heavy nuclei are considered to be static. Then, setting our wave vector along the z axis as $\vec{k} = k\,\hat{z}$, we can Fourier analyse the oscillating electron part and get a system describing our electron distribution to first order as

$$f = f_0(v^2, \theta_s) + \delta f \tag{3.1}$$

$$\delta f = \tilde{f}\, e^{i(kz-\omega t)}\,. \tag{3.2}$$

Using Gauss's and Faraday's laws

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{3.3}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.4}$$

we can see that our oscillating distribution function (3.2) gives rise to an oscillation of the electric field in the z direction only, which in term means that no oscillating magnetic field is created at all. Thus, considering an *applied* constant magnetic field $\vec{B}_0$ along the wave vector and Fourier analysing the oscillating electric field, we get

$$\vec{E} = \delta E\,\hat{z} \tag{3.5}$$

$$\vec{B} = B_0\,\hat{z} \tag{3.6}$$

$$\delta E = \tilde{E}\, e^{i(kz-\omega t)}\,. \tag{3.7}$$

We note that

$$\frac{\partial}{\partial \vec{x}} \to ik\,\hat{z} \tag{3.8}$$

$$\frac{\partial}{\partial t} \to -i\omega \tag{3.9}$$

when acting on the Fourier analysed variables.

Now linearizing the evolution equation (2.21) to first order gives us, moving terms proportional to f_0 to the right hand side and canceling the exponentials from the Fourier analysis (see Appendix A.1)

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{x}} - \omega_c \frac{\partial}{\partial \varphi_v} - \omega_{cg}\frac{\partial}{\partial \varphi_s}\right)\tilde{f} = \\
&= -\frac{q\tilde{E}}{m}\frac{\partial f_0}{\partial v_z} + \frac{2\mu v_\perp \tilde{E}}{\hbar c^2}(\cos\varphi_v\cos\varphi_s + \sin\varphi_v\sin\varphi_s)\frac{\partial f_0}{\partial \theta_s} \\
&\quad - \frac{ik\mu v_\perp \tilde{E}}{mc^2}\left(\sin\theta_s + \cos\theta_s\frac{\partial}{\partial\theta_s}\right) \\
&\qquad \times (\cos\varphi_v\sin\varphi_s - \sin\varphi_v\cos\varphi_s)\frac{\partial f_0}{\partial v_z},
\end{aligned} \tag{3.10}$$

where the *cyclotron frequency* $\omega_c = qB_0/m$, the *spin precession frequency* $\omega_{cg} = (g/2)\omega_c$ and the *electron gyromagnetic ratio* $g \approx 2.002319$ have been introduced. The velocity is given in cylindrical coordinates $\vec{v} = \vec{v}(v_\perp, \varphi_v, v_z)$ and the spin in spherical $\vec{s} = \vec{s}(s, \theta_s, \varphi_s)$. Since the considered particles are electrons, m denotes the *electron mass*, $\mu = (g/2)\mu_B$ the *magnetic moment for electrons*, where $\mu_B = q\hbar/2m$ is the *Bohr magneton*, and $q = -|e|$ the *electron charge*.

3.2 THE DISTRIBUTION FUNCTION

Making a series ansatz of the first order distribution function $\tilde{f}$ will let us handle most dependencies quite elegantly:

$$\tilde{f} = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g_{n,m}(v_z, v_\perp, \theta_s)\,\psi_n(v_\perp, \varphi_v)\,e^{im\varphi_s} \tag{3.11}$$

where n and m are integers. ψ_n both has the properties of orthonormality

$$\int_0^{2\pi} d\varphi_v\,\psi_{n_1}\psi_{n_2} = \begin{cases} 1, & \text{if } n_1 = n_2 \\ 0, & \text{if } n_1 \neq n_2 \end{cases} \tag{3.12}$$

and can be expanded into a series of Bessel functions $\mathcal{J}_{n'}$:

$$\begin{aligned}
\psi_n(v_\perp, \varphi_v) &= \frac{1}{\sqrt{2\pi}} e^{in\varphi_v - \frac{ik_\perp v_\perp}{\omega_v}\sin\varphi_v} \\
&= \frac{1}{\sqrt{2\pi}}\sum_{n'=-\infty}^{\infty}\mathcal{J}_{n'}\left(\frac{k_\perp v_\perp}{\omega_c}\right)e^{i(n-n')\varphi_v}
\end{aligned} \tag{3.13}$$

Generally this would mean that our solutions would contain Bessel functions, but since we are studying electrostatic Langmuir waves ($k_\perp = 0$) they will simplify quite a bit as only one term survives:

$$\mathcal{J}_{n'} = \begin{cases} 1\,, & \text{if } n' = 0 \\ 0\,, & \text{if } n' \neq 0 \end{cases} \tag{3.14}$$

Inserting (3.13) into (3.11) and performing the azimuthal angular derivatives on the left hand side of eq. (3.10) for this ansatz now just spits out the integers n and m respectively. To solve the equation we then multiply both sides by the complex conjugate $\psi^*_{n'}\, e^{-im'\varphi_s}/\sqrt{2\pi}$ and integrate over $\varphi_v\ \varphi_s$, orthonormality giving

$$\left(-i\omega + ikv_z - in\omega_c - im\omega_{cg}\right) g_{n,m} = \mathcal{I}_{n,m}. \tag{3.15}$$

$\mathcal{I}_{n,m}$ are integrals surviving from the right hand side of eq. (3.10) for different n and m:

$$\begin{aligned}\mathcal{I}_{n,m} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi_v \int_0^{2\pi} d\varphi_s\, e^{-in\varphi_v} e^{-im\varphi_s} \\ \times \Bigg[-\frac{q\tilde{E}}{m}\frac{\partial f_0}{\partial v_z} + \frac{2\mu v_\perp \tilde{E}}{\hbar c^2}(\cos\varphi_v\cos\varphi_s + \sin\varphi_v \sin\varphi_s)\frac{\partial f_0}{\partial \theta_s} \\ - \frac{ik\mu v_\perp \tilde{E}}{mc^2}\left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial\theta_s}\right) \\ \times\ (\cos\varphi_v \sin\varphi_s - \sin\varphi_v\cos\varphi_s)\frac{\partial f_0}{\partial v_z}\Bigg]\end{aligned} \tag{3.16}$$

We now use the following properties

$$\int_0^{2\pi} d\varphi\, e^{in\varphi} = \begin{cases} 2\pi, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases} \tag{3.17}$$

$$\int_0^{2\pi} d\varphi\, \cos\varphi\, e^{in\varphi} = \begin{cases} \pi, & \text{if } n = \pm 1 \\ 0, & \text{if } n \neq \pm 1 \end{cases} \tag{3.18}$$

$$\int_0^{2\pi} d\varphi\, \sin\varphi\, e^{in\varphi} = \begin{cases} \pm i\pi, & \text{if } n = \pm 1 \\ 0, & \text{if } n \neq \pm 1 \end{cases} \tag{3.19}$$

to evaluate the integrals and we then note that only three terms will survive (see Appendix A.2):

$$\mathcal{I}_{0,0} = -\frac{2\pi q\tilde{E}}{m}\frac{\partial f_0}{\partial v_z} \tag{3.20}$$

$$\begin{aligned}\mathcal{I}_{\pm1,\mp1} =\ & \pm\frac{\pi k v_\perp \mu \tilde{E}}{mc^2}\left(\sin\theta_s + \cos\theta_s\frac{\partial}{\partial\theta_s}\right)\frac{\partial f_0}{\partial v_z} \\ & + \frac{2\pi v_\perp \mu\tilde{E}}{\hbar c^2}\frac{\partial f_0}{\partial\theta_s}\end{aligned} \tag{3.21}$$

Using eq. (3.15) we see that

$$g_{n,m} = \frac{\mathcal{I}_{n,m}}{-i(\omega - kv_z + n\omega_c + m\omega_{cg})} \tag{3.22}$$

from which we can now finally by putting everything together calculate $\tilde{f}$ using the ansatz we made (3.11):

$$\begin{aligned}
\tilde{f} = &-\frac{iq\tilde{E}}{m}\frac{\partial f_0}{\partial v_z}\frac{1}{w - kv_z} \\
&+ \frac{ikv_\perp \mu \tilde{E}}{2mc^2}\left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_s}\right)\frac{\partial f_0}{\partial v_z} \\
&\times \left(\frac{e^{i\varphi_v}\, e^{-i\varphi_s}}{\omega - \Delta\omega_c - kv_z} - \frac{e^{-i\varphi_v}\, e^{i\varphi_s}}{\omega + \Delta\omega_c - kv_z}\right) \\
&+ \frac{iv_\perp \mu \tilde{E}}{\hbar c^2}\frac{\partial f_0}{\partial \theta_s} \\
&\times \left(\frac{e^{i\varphi_v}\, e^{-i\varphi_s}}{\omega - \Delta\omega_c - kv_z} + \frac{e^{-i\varphi_v}\, e^{i\varphi_s}}{\omega + \Delta\omega_c - kv_z}\right)
\end{aligned} \tag{3.23}$$

where the frequency $\Delta\omega_c = \omega_{cg} - \omega_c$ has been introduced.

3.3 CURRENT DENSITY

The total current density can now be calculated, using eqs. (3.23) and (2.21) (see Appendix A.3). Our total current density $\vec{j}_{\text{tot}}$ is becomes

$$\begin{aligned}
\vec{j}_{\text{tot}} = &-\frac{iq^2\tilde{E}}{m}\,\hat{z}\int d\Omega\, v_z \frac{\partial f_0/\partial v_z}{\omega - kv_z} \\
&+ \frac{i3k\mu^2\omega\tilde{E}}{4mc^4}\,\hat{z}\int d\Omega\, v_\perp^2 \sin\theta_s \left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial\theta_s}\right) \\
&\times \left(\frac{\partial f_0/\partial v_z}{\omega - \Delta\omega_c - kv_z} + \frac{\partial f_0/\partial v_z}{\omega + \Delta\omega_c - kv_z}\right) \\
&+ \frac{i3\mu^2\omega\tilde{E}}{2\hbar c^4}\,\hat{z}\int d\Omega\, v_\perp^2 \sin\theta_s \\
&\times \left(\frac{\partial f_0/\partial\theta_s}{\omega - \Delta\omega_c - kv_z} - \frac{\partial f_0/\partial\theta_s}{\omega + \Delta\omega_c - kv_z}\right),
\end{aligned} \tag{3.24}$$

where the first term corresponds to the classic free current and the other terms to the polarization current, the magnetization current gives no contribution at all.

If we would not just be dealing with electrostatic waves the current density would have parts in the perpendicular x and y directions, with all parts containing Bessel function terms.

3.4 THE GENERAL DISPERSION RELATION

Now, knowing that our constant magnetic field has no curl and the Fourier analysis result $(\partial/\partial t)\vec{E} = -i\omega\tilde{E}\,\hat{z}$, Ampere's law (2.15) becomes

$$0 = \mu_0\,\vec{j}_{\text{tot}} - \frac{i\omega}{c^2}\tilde{E}\,\hat{z}. \tag{3.25}$$

Inserting our current density (3.24), we note that all terms point along the z axis and contain a factor $i\tilde{E}$, thus rearranging the expression we get that

$$\begin{aligned} 0 = &-\omega - \omega_p^2 \int d\Omega\, v_z \frac{\partial\hat{f}_0/\partial v_z}{\omega - kv_z} \\ &+ \frac{3k\hbar^2\omega_p^2\omega}{16m^2c^4}\int d\Omega\, v_\perp^2 \sin\theta_s \left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial\theta_s}\right) \\ &\quad\times\left(\frac{\partial\hat{f}_0/\partial v_z}{\omega - \Delta\omega_c - kv_z} + \frac{\partial\hat{f}_0/\partial v_z}{\omega + \Delta\omega_c - kv_z}\right) \\ &+ \frac{3\hbar\omega_p^2\omega}{8mc^4}\int d\Omega\, v_\perp^2 \sin\theta_s \\ &\quad\times\left(\frac{\partial\hat{f}_0/\partial\theta_s}{\omega - \Delta\omega_c - kv_z} - \frac{\partial\hat{f}_0/\partial\theta_s}{\omega + \Delta\omega_c - kv_z}\right), \end{aligned} \tag{3.26}$$

where $\hat{f}_0 = f_0/n_0$ has been introduced to rescale the distribution function, $\omega_p^2 = q^2 n_0/\epsilon_0 m$ is the electron plasma frequency and n_0 is the total number of particles. This is our general dispersion relation for Langmuir waves with spin-orbit coupling, leaving us three integrals to solve for specific distribution functions f_0 and studied approximations, with all spin dependence contained in the second and third integrals.

We can now presume that we will find wave modes when $\omega \sim \Delta\omega_c$ since the last terms will be large at that point for small k.

4 ANALYSING THE DISPERSION RELATION

To analyse the dispersion relation we will need to solve the involved integrals, whom are complicated a bit by the included denominators involving v_z. We will be studying a distribution function $f_0 = f_0(v^2, \theta_s)$ where the velocity dependent part takes the form of an ordinary Maxwellian distribution and the spin dependent part takes a convenient form [9],

$$f_0(v^2, \theta_s) = \frac{n_0}{N_\mathrm{M}} e^{-\frac{v^2}{v_t^2}} \times \frac{1}{N_\mathrm{S}} \left[e^{\frac{\mu B_0}{k_B T}} (1 + \cos\theta_s) + e^{-\frac{\mu B_0}{k_B T}} (1 - \cos\theta_s) \right] \tag{4.1}$$

where the prefactors are given to normalize the distribution as

$$\frac{1}{N_\mathrm{M}} = \left(\frac{1}{\pi v_t^2} \right)^{3/2} \tag{4.2}$$

$$\frac{1}{N_\mathrm{S}} = \frac{1}{4\pi \cosh\left(\frac{\mu B_0}{k_B T} \right)} \tag{4.3}$$

for the Maxwellian and spin part respectively, T is the temperature, k_B is the Boltzmann constant and $v_t = \sqrt{2 k_B T / m}$ is the thermal velocity. To simplify the integrals we will consider the situation when the frequency is close to resonance with $\Delta\omega_c$,

$$\omega \sim \Delta\omega_c \tag{4.4}$$

thus neglecting the terms with denominators $1/(\omega + \Delta\omega_c - k v_z)$ in (3.26) since they will be small compared to the terms with denominators $1/(\omega - \Delta\omega_c - k v_z)$.

4.1 HIGH-FREQUENCY LIMIT

The high frequency limit is in our case as

$$k v_z \ll \omega - \Delta\omega_c\,, \tag{4.5}$$

letting us expand the denominators of the dispersion relation (3.26) and, for the moment neglecting the poles in $\omega = k v_z$ and

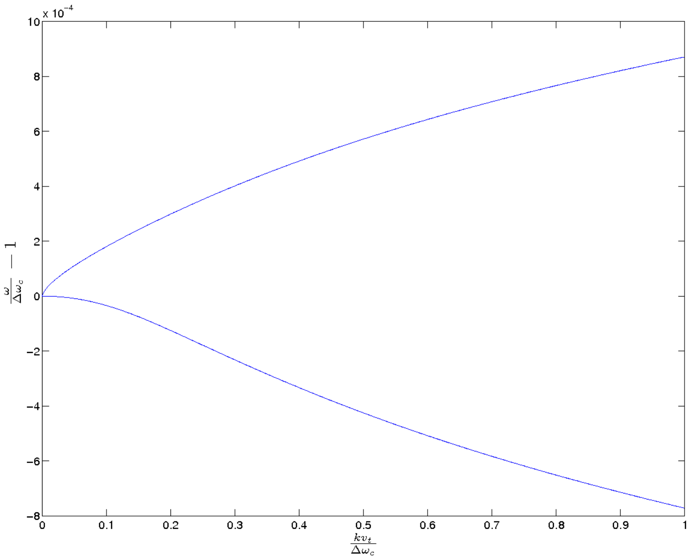

Figure 1: Two new wave modes appear close to the resonant frequency. Plotted for astrophysical values:
$n_0 = 10^{21}\,\mathrm{cm}^{-3}$, $T = 10^{10}\,\mathrm{K}$ and $B_0 = 6 \times 10^6\,\mathrm{T}$

$\omega - \Delta\omega_c = kv_z$ which will be expanded upon in the next section, solve the relation to (see Appendix A.4)

$$\begin{aligned}&\omega^2 \left\{ 1 + \frac{\hbar^2\omega_p^2}{4m^2c^4} \left[\frac{k^2v_t^2}{(\omega - \Delta\omega_c)^2} + \frac{3k^4v_t^4}{2(\omega - \Delta\omega_c)^4} \right] \right. \\ &\left. \quad + \frac{\hbar^2\omega_p^2}{4m^2c^4} \frac{mv_t^2}{2\hbar} \tanh\left(\frac{\mu B_0}{k_B T}\right) \left[\frac{2}{(\omega - \Delta\omega_c)} + \frac{k^2v_t^2}{(\omega - \Delta\omega_c)^3} \right] \right\} \\ &= \omega_p^2 + \frac{3\omega_p^2 k^2 v_t^2}{2\omega^2} . \end{aligned} \tag{4.6}$$

This relation gives rise to two new wave modes arising from the resonant frequency $\Delta\omega_c$ in addition to the standard mode for classical Langmuir waves. See Figure 1. We can see that frequencies need to be very close to the resonance in order for the modes to be found: For astrophysical applications the difference from resonance is on the order of 10^{-4} for variations on the scale of $kv_t \sim \Delta\omega_c$.

4.2 LANDAU DAMPING

When calculating the integrals in the previous section the singularities at $v_z = (\omega - \Delta\omega_c)/k$ and $v_z = \omega/k$ where neglected as the expansion was performed. In reality the solution is not that simple as the singularities contribute to an effect named after its discoverer, Landau damping, where particles may absorb energy from, or lose energy to, resonant electromagnetic waves

[3, 6]. As an aside, the resonant wave-particle process that we are interested in may resemble *cyclotron damping* even more than Landau damping. From a mathematical point of view, however, all resonant wave-particle processes are quite similar within linear theory, hence we will not consider the distinction between the two in this thesis.

To treat the problem correctly one has to assume that ω has an imaginary part in addition to the real, $\omega = \omega_r - i\omega_i$, noting that for positive frequencies ω_i the oscillations in eq. (3.2) will decrease with time. This imaginary part means that our line integral over the v_z axis has to be turned into a contour integral in the complex plane. This is a slight oversimplification, but it is sufficient for our purposes. See e. g. [8] for a more detailed discussion on the issues of Landau damping.

For a considered small damping term and using methods from standard residue theory, [6, 7] the integrals in our dispersion relation (3.26) are calculated by using

$$\int_{-\infty}^{\infty} dz \, \frac{\partial f/\partial z}{z-a} = \text{p.v.} \int_{-\infty}^{\infty} dz \, \frac{\partial f/\partial z}{z-a} + i\pi \left(\frac{\partial f}{\partial z} \right) \bigg|_{z=a} , \quad (4.7)$$

where *p.v.* denotes the principal value of the integral and the imaginary contribution comes from integrating around the singularity in a semi-circle. Still studying the same high frequency limit and resonance the principal values of our integrals are as already calculated without considering the singularity, as done in the previous section ending up with eq. (4.6). The remaining imaginary terms gives contributions as, only considering the integrals over v_z,

$$-\frac{i\pi}{k} \left(v_z \frac{\partial e^{-\frac{v_z^2}{v_t^2}}}{\partial v_z} \right) \Bigg|_{v_z = \frac{\omega_r}{k}} = -\frac{2i\pi\omega_r^2}{k^3 v_t^2} \, e^{-\frac{\omega_r^2}{k^2 v_t^2}} \quad (4.8)$$

$$-\frac{i\pi}{k} \left(\frac{\partial e^{-\frac{v_z^2}{v_t^2}}}{\partial v_z} \right) \Bigg|_{v_z = \frac{\omega_r \mp \Delta\omega_c}{k}} = -\frac{2i\pi(\omega_r \mp \Delta\omega_c)}{k^2 v_t^2} \, e^{-\frac{(\omega_r \mp \Delta\omega_c)^2}{k^2 v_t^2}} \quad (4.9)$$

$$-\frac{i\pi}{k} \left(e^{-\frac{v_z^2}{v_t^2}} \right) \Bigg|_{v_z = \frac{\omega_r \mp \Delta\omega_c}{k}} = -\frac{i\pi}{k} \, e^{-\frac{(\omega_r \mp \Delta\omega_c)^2}{k^2 v_t^2}} . \quad (4.10)$$

As $\omega_r \sim \Delta\omega_c$ we can once again ignore the terms proportional to $\exp[-(\omega_r + \Delta\omega_c)/(k^2 v_t^2)]$.

Yet another imaginary part comes from explicitly inserting $\omega = \omega_r - i\omega_i$ in in the dispersion relation (3.26). Finally solving for ω_i [6] gives the damping term: For our purposes it will be sufficient to do this with keeping only the terms to first order

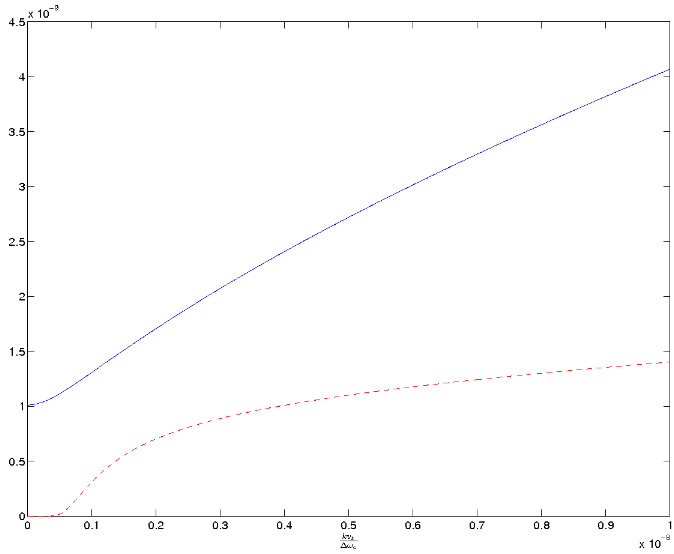

Figure 2: The upper wave mode $\omega_r/\Delta\omega_c - 1$ (solid line) plotted against the corresponding imaginary frequency $\omega_i/\Delta\omega_c$ (dashed line). Plotted for astrophysical values:
$n_0 = 10^{21}\,\mathrm{cm}^{-3}$, $T = 10^{10}\,\mathrm{K}$ and $B_0 = 6\times 10^{6}\,\mathrm{T}$

from the principal values on the left and the largest residue term on the right hand side,

$$\begin{aligned}(\omega_r - i\omega_i)^2 + \frac{\hbar^2\omega_p^2}{4m^2c^4}\frac{mv_t^2}{\hbar}\tanh\left(\frac{\mu B_0}{k_B T}\right)\frac{(\omega_r - i\omega_i)^2}{\omega_r - i\omega_i - \Delta\omega_c} = \\ = -i\frac{\hbar^2\omega_p^2}{4m^2c^4}\frac{mv_t^2}{\hbar}\tanh\left(\frac{\mu B_0}{k_B T}\right)\frac{(\omega_r - i\omega_i)^2}{kv_t}\,e^{-\left(\frac{\omega_r - i\omega_i - \Delta\omega_c}{kv_t}\right)^2}.\end{aligned} \tag{4.11}$$

Ignoring the very small terms of ω_i^2, this is solved to

$$\omega_i = \frac{\frac{\hbar^2\omega_p^2}{4m^2c^4}\frac{mv_t^2}{\hbar}\tanh\left(\frac{\mu B_0}{k_B T}\right)\Delta\omega_c\frac{\omega_r}{kv_t}\,e^{-\left(\frac{\omega_r-\Delta\omega_c}{kv_t}\right)^2}}{\left\{1 + \frac{\hbar^2\omega_p^2}{4m^2c^4}\frac{mv_t^2}{\hbar}\tanh\left(\frac{\mu B_0}{k_B T}\right)\Delta\omega_c\left[\frac{\omega_r - 2\Delta\omega_c}{(\omega_r-\Delta\omega_c)^2}\right]\right\}} \tag{4.12}$$

which is our damping term to first order. For the considered limit the damping is very small, only being of the same order as the difference between the wave modes and the resonance at very small values of k. See Figure 2.

4.3 SUMMARY

The modified kinetic theory including the spin-orbit terms and the polarization current gives rise to two new wave modes close to the resonance $\omega \sim \Delta\omega_c$. For small wave numbers the modes are very close to the resonance even for astrophysical applications, allowing deviations on the order of $\omega \sim \Delta\omega_c \pm 10^{-4}$ for variations on the scale of $kv_t \sim \Delta\omega_c$. These additional wave modes are very weakly damped.

A
CALCULATIONS

This appendix contains some calculations omitted in the main text.

A.1 LINEARIZING THE EVOLUTION EQUATION

To linearize the evolution equation (2.13) we insert the chosen forms of our variables (3.1), (3.5) and (3.6)

$$\begin{aligned} f &= f_0(v^2, \theta_s) + \delta f \\ \vec{E} &= \delta E\, \hat{z} \\ \vec{B} &= B_0\, \hat{z}\,, \end{aligned}$$

into it and study terms of total first order. Keeping terms proportional to δf on the left hand side and moving terms proportional to f_0 to the right hand side, we get

$$\begin{aligned} &\frac{\partial\, \delta f}{\partial t} + \vec{v} \cdot \frac{\partial\, \delta f}{\partial \vec{x}} + \left[\frac{q}{m} \vec{v} \times \vec{B}_0 + \frac{\mu}{m} \frac{\partial}{\partial \vec{x}} (\vec{s} \cdot \vec{B}_0) \right] \cdot \frac{\partial\, \delta f}{\partial \vec{v}} \\ &+ \frac{2\mu}{\hbar} (\vec{s} \times \vec{B}_0) \cdot \frac{\partial\, \delta f}{\partial \vec{s}} + \frac{\mu}{m} \frac{\partial}{\partial \vec{x}} \cdot \left[\left(\vec{B}_0 \cdot \frac{\partial}{\partial \vec{s}} \right) \frac{\partial}{\partial \vec{v}} \right] \delta f = \\ = &- \left\{ \frac{q\, \delta E}{m}\, \hat{z} - \frac{\mu}{mc^2} \frac{\partial}{\partial \vec{x}} \left[\vec{s} \cdot (\vec{v} \times \delta E\, \hat{z}) \right] \right\} \cdot \frac{\partial f_0}{\partial \vec{v}} \\ &- \frac{2\mu}{\hbar c^2} \left[\vec{s} \times (\vec{v} \times \delta E\, \hat{z}) \right] \cdot \frac{\partial f_0}{\partial \vec{s}} \\ &- \frac{\mu}{mc^2} \frac{\partial}{\partial \vec{x}} \cdot \left\{ \left[(\vec{v} \times \delta E\, \hat{z}) \cdot \frac{\partial}{\partial \vec{s}} \right] \frac{\partial}{\partial \vec{v}} \right\} f_0\,, \end{aligned} \tag{A.1}$$

which then is our first order linearized evolution equation. Using cylindrical coordinates for the velocity ($\vec{v} = \vec{v}(v_\perp, \varphi_v, v_z)$) and spherical for the semiclassical spin vector ($\vec{s} = \vec{s}(\theta_s, \varphi_s)$)

$$\vec{v} = \hat{x}\, v_\perp \cos\varphi_v + \hat{y}\, v_\perp \sin\varphi_v + \hat{z}\, v_z \tag{A.2}$$

$$\vec{s} = \hat{x}\, \sin\theta_s \cos\varphi_s + \hat{y}\, \sin\theta_s \sin\varphi_s + \hat{z}\, \cos\theta_s \tag{A.3}$$

most of the terms in the linearized equation can be calculated. Keeping in mind our Fourier analysis (3.2) and (3.7)

$$\begin{aligned} \delta f &= \tilde{f}\, e^{i(kz-\omega t)} \\ \delta E &= \tilde{E}\, e^{i(kz-\omega t)} \end{aligned}$$

and the accompanying properties (3.8)–(3.9)

$$\begin{aligned} \frac{\partial}{\partial \vec{x}} &\to ik\, \hat{z} \\ \frac{\partial}{\partial t} &\to -i\omega \end{aligned}$$

when acting on oscillating exponentials, we can from the left hand side of (A.1) calculate

$$\frac{\partial}{\partial \vec{x}}(\vec{s} \cdot \vec{B}_0) = 0 \tag{A.4}$$

$$\vec{v} \times \vec{B}_0 = -B_0 v_\perp \hat{\varphi}_v \tag{A.5}$$

$$\vec{s} \times \vec{B}_0 = -\sin\theta_s\,\hat{\varphi}_s \tag{A.6}$$

$$\frac{\partial}{\partial \vec{x}} \cdot \left[\left(\vec{B}_0 \cdot \frac{\partial}{\partial \vec{s}}\right)\frac{\partial}{\partial \vec{v}}\right] = 0 \tag{A.7}$$

since the magnetic field is constant. Similarly, from the right hand side we can calculate

$$\delta E\,\hat{z} \cdot \frac{\partial}{\partial \vec{v}} = \delta E \frac{\partial}{\partial v_z} \tag{A.8}$$

$$\begin{aligned}\frac{\partial}{\partial \vec{x}}\big[\vec{s} \cdot (\vec{v} \times \delta E\,\hat{z})\big] \cdot \frac{\partial}{\partial \vec{v}} &= -ikv_\perp \delta E \sin\theta_s \\ &\quad \times (\cos\varphi_v \sin\varphi_s - \sin\varphi_v \cos\varphi_s)\frac{\partial}{\partial v_z}\end{aligned} \tag{A.9}$$

$$\begin{aligned}\frac{\partial}{\partial \vec{x}} \cdot \left[(\vec{v} \times \delta E\,\hat{z}) \cdot \frac{\partial}{\partial \vec{s}}\right]\frac{\partial}{\partial \vec{v}} &= -ikv_\perp \delta E \cos\theta_s \\ &\quad \times (\cos\varphi_v \sin\varphi_s - \sin\varphi_v \cos\varphi_s)\frac{\partial}{\partial \theta_s}\frac{\partial}{\partial v_z}\end{aligned} \tag{A.10}$$

$$\begin{aligned}\big[\vec{s} \times (\vec{v} \times \delta E\,\hat{z})\big] \cdot \frac{\partial}{\partial \vec{s}} &= v_\perp \delta E \\ &\quad \times (\cos\varphi_v \cos\varphi_s + \sin\varphi_v \sin\varphi_s)\frac{\partial}{\partial \theta_s}\,.\end{aligned} \tag{A.11}$$

Collecting all terms, this gives us a linearized and Fourier analysed evolution equation as

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} - \omega_c \frac{\partial}{\partial \varphi_v} - \omega_{cg}\frac{\partial}{\partial \varphi_s}\right)\delta f = \\ &= -\frac{q\delta E}{m}\frac{\partial f_0}{\partial v_z} \\ &\quad + \frac{2\mu v_\perp \delta E}{\hbar c^2}(\cos\varphi_v \cos\varphi_s + \sin\varphi_v \sin\varphi_s)\frac{\partial f_0}{\partial \theta_s} \\ &\quad - \frac{ik\mu v_\perp \delta E}{mc^2}\left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_s}\right) \\ &\quad \times (\cos\varphi_v \sin\varphi_s - \sin\varphi_v \cos\varphi_s)\frac{\partial f_0}{\partial v_z}\,,\end{aligned} \tag{A.12}$$

where the cyclotron frequency $\omega_c = qB_0/m$, the spin precession frequency $\omega_{cg} = (g/2)\omega_c$ and the electron gyromagnetic ratio g have been introduced. Canceling the exponentials from the Fourier analysis gives us the final expression in our calculations (3.10).

A.2 AZIMUTHAL ANGULAR INTEGRALS $\mathcal{I}_{n,m}$

To solve the angular integrals (3.16)

$$
\begin{aligned}
\mathcal{I}_{n,m} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi_v \int_0^{2\pi} d\varphi_s \, e^{-in\varphi_v} \, e^{-im\varphi_s} \\
\times \Bigg[-\frac{q\tilde{E}}{m} \frac{\partial f_0}{\partial v_z} + \frac{2\mu v_\perp \tilde{E}}{\hbar c^2} (\cos\varphi_v \cos\varphi_s + \sin\varphi_v \sin\varphi_s) \frac{\partial f_0}{\partial \theta_s} \\
- \frac{ik\mu v_\perp \tilde{E}}{mc^2} \left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_s} \right) \\
\times \, (\cos\varphi_v \sin\varphi_s - \sin\varphi_v \cos\varphi_s) \frac{\partial f_0}{\partial v_z} \Bigg]
\end{aligned}
$$

the properties (3.17)–(3.19)

$$
\begin{aligned}
\int_0^{2\pi} d\varphi \, e^{in\varphi} &= \begin{cases} 2\pi, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases} \\
\int_0^{2\pi} d\varphi \, \cos\varphi \, e^{in\varphi} &= \begin{cases} \pi, & \text{if } n = \pm 1 \\ 0, & \text{if } n \neq \pm 1 \end{cases} \\
\int_0^{2\pi} d\varphi \, \sin\varphi \, e^{in\varphi} &= \begin{cases} \pm i\pi, & \text{if } n = \pm 1 \\ 0, & \text{if } n \neq \pm 1 \end{cases}
\end{aligned}
$$

will be used: We note that since all terms contain the exponentials only terms with $n, m = 0, \pm 1$ will survive at all. The integrals surviving are

$$
\begin{aligned}
\mathcal{I}_{0,0} &= -\frac{q\tilde{E}}{2\pi m} \frac{\partial f_0}{\partial v_z} \int_0^{2\pi} d\varphi_v \int_0^{2\pi} d\varphi_s \\
&= -\frac{q\tilde{E}}{2\pi m} \frac{\partial f_0}{\partial v_z} \cdot 4\pi^2
\end{aligned} \tag{A.13}
$$

$$
\begin{aligned}
\mathcal{I}_{\pm 1, \mp 1} = & -\frac{ikv_\perp \mu \tilde{E}}{2\pi mc^2} \left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_s} \right) \frac{\partial f_0}{\partial v_z} \\
& \times \int_0^{2\pi} d\varphi_v \int_0^{2\pi} d\varphi_s \, (\cos\varphi_v \sin\varphi_s - \sin\varphi_v \cos\varphi_s) \, e^{\mp i\varphi_v} \, e^{\pm i\varphi_s} \\
& + \frac{v_\perp \mu \tilde{E}}{\pi \hbar c^2} \frac{\partial f_0}{\partial \theta_s} \int_0^{2\pi} d\varphi_v \int_0^{2\pi} d\varphi_s \, (\cos\varphi_v \cos\varphi_s + \sin\varphi_v \sin\varphi_s) \, e^{\mp i\varphi_v} \, e^{\pm i\varphi_s} \\
= & -\frac{ikv_\perp \mu \tilde{E}}{2\pi mc^2} \left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial \theta_s} \right) \frac{\partial f_0}{\partial v_z} \cdot (\pm i 2\pi^2) \\
& + \frac{2v_\perp \mu \tilde{E}}{2\pi \hbar c^2} \frac{\partial f_0}{\partial \theta_s} \cdot 2\pi^2 \, ,
\end{aligned} \tag{A.14}
$$

which are simplified into the expressions used in the main text (3.20)–(3.21).

A.3 CURRENT DENSITY INTEGRALS

We are looking to calculate the total current density $\vec{j}_{\text{tot}}$ (2.21)

$$\vec{j}_{\text{tot}} = q \int d\Omega\, \vec{v}\, f(\vec{x},\vec{v},\vec{s},t) + \nabla \times 3\mu \int d\Omega\, \vec{s}\, f(\vec{x},\vec{v},\vec{s},t) - \frac{3\mu}{c^2}\frac{\partial}{\partial t}\int d\Omega\, \vec{v} \times \vec{s}\, f(\vec{x},\vec{v},\vec{s},t)$$

using our distribution function f. However, since the unperturbed distribution function f_0 will not give any contribution we will only consider the perturbed $\tilde{f}$ (3.23)

$$\begin{aligned}\tilde{f} = &-\frac{iq\tilde{E}}{m}\frac{\partial f_0/\partial v_z}{w - kv_z} \\ &+ \frac{ikv_\perp \mu \tilde{E}}{2mc^2}\left(\sin\theta_s + \cos\theta_s \frac{\partial}{\partial\theta_s}\right)\frac{\partial f_0}{\partial v_z} \\ &\times \left(\frac{e^{i\varphi_v}\, e^{-i\varphi_s}}{\omega - \Delta\omega_c - kv_z} - \frac{e^{-i\varphi_v}\, e^{i\varphi_s}}{\omega + \Delta\omega_c - kv_z}\right) \\ &+ \frac{iv_\perp \mu \tilde{E}}{\hbar c^2}\frac{\partial f_0}{\partial\theta_s}\left(\frac{e^{i\varphi_v}\, e^{-i\varphi_s}}{\omega - \Delta\omega_c - kv_z} + \frac{e^{-i\varphi_v}\, e^{i\varphi_s}}{\omega + \Delta\omega_c - kv_z}\right).\end{aligned}$$

The first term in our total current density, corresponding to the free current density, is calculated by first noting that since the velocity does not contain any spin-dependence the exponentials over the azimuthal angles in spin-space $\exp(\pm i\varphi_s)$ in $\tilde{f}$ are integrated to zero, nullifying all terms except one to solve. Again using cylindrical coordinates for the velocity (A.2), we also note that the trigonometric functions in velocity-space integrate to zero. The free current density thus becomes

$$\vec{j}_{\text{F}} = -\frac{iq^2\tilde{E}}{m}\hat{z}\int d\Omega\, v_z \frac{\partial f_0/\partial v_z}{\omega - kv_z}. \tag{A.15}$$

Likewise the second term in the total current density, corresponding to the magnetization current $\vec{j}_{\text{M}}$, will give no contribution, since the spin does not contain any velocity dependence and thus the corresponding exponentials for azimuthal angles in velocity-space integrates to zero and the remaining term simply becomes, keeping the Fourier analysis result $\partial/\partial\vec{x} \to ik\,\hat{z}$ (3.8) in mind,

$$\vec{j}_{\text{M}} = -\frac{3iq\mu\tilde{E}}{m}\cdot ik\,\hat{z}\times\hat{z}\int d\Omega\, \cos\theta_s \frac{\partial f_0/\partial v_z}{\omega - kv_z} = 0\,, \tag{A.16}$$

where spherical coordinates once again have been used for the semi-classical spin vector (A.3). Finally the third term, corresponding to a polarization current density $\vec{j}_{\text{P}}$, is handled by noting that only the z part of $\vec{v}\times\vec{s}$,

$$\begin{aligned}\vec{v}\times\vec{s} = &\;\hat{x}\,(v_\perp \sin\varphi_v \cos\theta_s - v_z \sin\varphi_s \sin\theta_s) \\ &+ \hat{y}\,(v_z \cos\varphi_s \sin\theta_s - v_\perp \cos\varphi_v \cos\theta_s) \\ &+ \hat{z}\, v_\perp \sin\theta_s(\cos\varphi_v \sin\varphi_s - \sin\varphi_v \cos\varphi_s)\end{aligned} \tag{A.17}$$

contains the necessary spin and velocity azimuthal angle dependence to survive integration for all terms of $\tilde{f}$: Note also that as the first term in $\tilde{f}$ does not contain any exponentials it will also integrate to zero for all terms. This gives us the polarization current density like

$$\begin{aligned}\vec{j}_{\mathrm{P}} = &-\frac{3\mu}{c^2}\frac{\partial}{\partial t}\,\hat{z}\int d\Omega\, v_\perp \sin\theta_s(\cos\varphi_v \sin\varphi_s - \sin\varphi_v\cos\varphi_s)\\ &\times\left\{\frac{ikv_\perp \mu\tilde{E}}{2mc^2}\left(\sin\theta_s + \cos\theta_s\frac{\partial}{\partial\theta_s}\right)\frac{\partial f_0}{\partial v_z}\right.\\ &\quad\times\left[\frac{e^{i\varphi_v}\,e^{-i\varphi_s}}{(\omega-\Delta\omega_c-kv_z)} - \frac{e^{-i\varphi_v}\,e^{i\varphi_s}}{(\omega+\Delta\omega_c-kv_z)}\right]\\ &\quad\left.+\frac{iv_\perp\mu\tilde{E}}{\hbar c^2}\frac{\partial f_0}{\partial\theta_s}\left[\frac{e^{i\varphi_v}\,e^{-i\varphi_s}}{(\omega-\Delta\omega_c-kv_z)} + \frac{e^{-i\varphi_v}\,e^{i\varphi_s}}{(\omega+\Delta\omega_c-kv_z)}\right]\right\}.\end{aligned} \tag{A.18}$$

Again using a result from the Fourier analysis $\partial/\partial t \to -i\omega$ (3.9) and solving the integrals using (3.17)–(3.19) (as used in the previous section), we get

$$\begin{aligned}\vec{j}_{\mathrm{P}} = &\frac{i3k\mu^2\omega\tilde{E}}{4mc^4}\,\hat{z}\int d\Omega\, v_\perp^2\sin\theta_s\left(\sin\theta_s+\cos\theta_s\frac{\partial}{\partial\theta_s}\right)\\ &\times\left(\frac{\partial f_0/\partial v_z}{\omega-\Delta\omega_c-kv_z}+\frac{\partial f_0/\partial v_z}{\omega+\Delta\omega_c-kv_z}\right)\\ &+\frac{i3\mu^2\omega\tilde{E}}{2\hbar c^4}\,\hat{z}\int d\Omega\, v_\perp^2\sin\theta_s\\ &\times\left(\frac{\partial f_0/\partial\theta_s}{\omega-\Delta\omega_c-kv_z}-\frac{\partial f_0/\partial\theta_s}{\omega+\Delta\omega_c-kv_z}\right)\end{aligned} \tag{A.19}$$

where the integration element $d\Omega$ has been kept for simplicity, a normalizing factor of $1/4\pi^2$ multiplied into the equation after the integration over the azimuthal angles have been performed.

A.4 SOLVING THE DISPERSION RELATION

We want to solve the general dispersion relation for our Langmuir waves (3.26)

$$\begin{aligned}0 = &-\omega-\omega_p^2\int d\Omega\, v_z\frac{\partial\hat{f}_0/\partial v_z}{\omega-kv_z}\\ &+\frac{3k\hbar^2\omega_p^2\omega}{16m^2c^4}\int d\Omega\, v_\perp^2\sin\theta_s\left(\sin\theta_s+\cos\theta_s\frac{\partial}{\partial\theta_s}\right)\\ &\times\left(\frac{\partial\hat{f}_0/\partial v_z}{\omega-\Delta\omega_c-kv_z}+\frac{\partial\hat{f}_0/\partial v_z}{\omega+\Delta\omega_c-kv_z}\right)\\ &+\frac{3\hbar\omega_p^2\omega}{8mc^4}\int d\Omega\, v_\perp^2\sin\theta_s\\ &\times\left(\frac{\partial\hat{f}_0/\partial\theta_s}{\omega-\Delta\omega_c-kv_z}-\frac{\partial\hat{f}_0/\partial\theta_s}{\omega+\Delta\omega_c-kv_z}\right),\end{aligned}$$

in the limit $kv_z \ll \omega - \Delta\omega_c$ for an unperturbed distribution function (4.1)

$$f_0(v^2, \theta_s) = \frac{n_0}{N_\mathrm{M}} e^{-\frac{v^2}{v_t^2}} \times \frac{1}{N_\mathrm{S}} \left[e^{\frac{\mu B_0}{k_B T}} (1 + \cos\theta_s) + e^{-\frac{\mu B_0}{k_B T}} (1 - \cos\theta_s) \right]$$

and $\hat{f}_0 = f_0/n_0$. To do this we first consider the resonance $\omega \sim \Delta\omega_c$, which is where we presume new wave modes to appear. This allows us to ignore the terms containing $1/(\omega + \Delta\omega_c - kv_z)$ since they will be comparably small. We will now expand the remaining denominators as

$$\frac{1}{\omega - kv_z} = \frac{1}{\omega} + \frac{kv_z}{\omega^2} + \frac{k^2 v_z^2}{\omega^3} + \frac{k^3 v_z^3}{\omega^4} + \dots \tag{A.20}$$

$$\begin{aligned} \frac{1}{\omega - \Delta\omega_c - kv_z} = {} & \frac{1}{\omega - \Delta\omega_c} + \frac{kv_z}{(\omega - \Delta\omega_c)^2} \\ & + \frac{k^2 v_z^2}{(\omega - \Delta\omega_c)^3} + \frac{k^3 v_z^3}{(\omega - \Delta\omega_c)^4} + \dots, \end{aligned} \tag{A.21}$$

cutting the expansions at fourth order.

To solve the integrals from here, we first note that integration over v_z will integrate odd functions in that space to zero. As f_0 is an even function of v_z and $\partial f_0/\partial v_z = -2v_z f_0/v_t^2$ is odd, different order terms of the expansion will survive in different terms of the integral. Solving all integrals after this expansion and simplification now gets us

$$\begin{aligned} 0 = {} & -\omega + \frac{\omega_p^2}{\omega} + \frac{3\omega_p^2 k^2 v_t^2}{2\omega^3} \\ & - \frac{\hbar^2 \omega_p^2 \omega}{4m^2 c^4} \left[\frac{k^2 v_t^2}{(\omega - \Delta\omega_c)^2} + \frac{3k^4 v_t^4}{2(\omega - \Delta\omega_c)^4} \right] \\ & - \frac{\hbar^2 \omega_p^2}{4m^2 c^4} \frac{m v_t^2 \omega}{2\hbar} \tanh\left(\frac{\mu B_0}{k_B T} \right) \left[\frac{2}{(\omega - \Delta\omega_c)} + \frac{k^2 v_t^2}{(\omega - \Delta\omega_c)^3} \right]. \end{aligned} \tag{A.22}$$

Rearranging the expression we get the dispersion relation as in eq. (4.6).

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